JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **206,** 611–622 (1997)
ARTICLE NO. AY975258

NOTE

On Some Integral Inequalities and Their Applications

Lechosław Hącia*

Submitted by William F. Ames

Received November 13, 1995

Some variants of two-dimensional integral inequalities, so-called inequalities of the Volterra–Fredholm type, are considered. In particular, generalizations of the Gronwall inequality are obtained. These results are applied to study the boundedness, stability and uniqueness of the solutions of some integral equations and their systems. © 1997 Academic Press

1. INTRODUCTION

Many authors have studied generalizations of the Gronwall inequality in two independent variables (see [1–3]). In this paper [4] special cases of two-dimensional inequalities of the Volterra type

$$u(x,y) \le f(x,y) + \int_0^x \int_0^y k(x,y,s,t) u(s,t)\, ds\, dt \tag{1}$$

were considered. Presented results were the generalization of the results of papers [1–3].

Our aim is to present various variants of integral inequalities of the Volterra–Fredholm type

$$u(x,t) \le f(x,t) + \int_0^t \int_a^b k(x,t,y,s) u(y,s)\, dy\, ds. \tag{2}$$

In particular, the Gronwall-type inequalities are obtained for (2). The studied results are applied in various problems of integral equations and partial differential equations of parabolic type.

* E-mail address: lhacia@abacus.math.put.poznan.pl.

0022-247X/97 $25.00

2. MAIN RESULTS

Using the theory of Volterra–Fredholm equations (see [5]), we can prove the following result on integral inequalities.

THEOREM 1. *Let f be a continuous function in $D = \{(x,t)\colon a \le x \le b, t \ge 0\}$ and k be nonnegative and continuous in $\Omega = \{(x,y,s,t)\colon a \le x, y \le b, 0 \le s \le t < \infty\}$. If the continuous function u satisfies inequality* (2), *then*

$$u(x,t) \le f(x,t) + \int_0^t \int_a^b r(x,t,y,s) f(y,s)\, dy\, ds, \tag{3}$$

where the resolvent kernel k is of the form

$$r(x,y,s,t) = \sum_{n=0}^{\infty} k_n(x,t,y,s) \tag{4}$$

and the iterated kernel k_n is defined by the following formula:

$$k_n(x,t,y,s) = \int_s^t \int_a^b k(x,t,p,q) k_{n-1}(p,q,y,s)\, dp\, dq,$$
$$n = 1,2,3,\dots,\ k_0(x,t,y,s) = k(x,t,y,s). \tag{5}$$

Proof. For a continuous and nonnegative function g in D, from inequality (2) we get the Volterra–Fredholm integral equation

$$u(x,t) = f(x,t) - g(x,t) + \int_0^t \int_a^b k(x,t,y,s) u(y,s)\, dy\, ds.$$

Using the resolvent method, we have

$$u(x,t) = f(x,t) - g(x,t) + \int_0^t \int_a^b r(x,t,y,s)[f(y,s) - g(y,s)]\, dy\, ds.$$

Since $g(x,t) \ge 0$, we obtain (3). ∎

Next, let us consider a special case of inequality (2) with $k(x,t,y,s) = A(x,t)B(y,s)$.

THEOREM 2. *Let A, B, f, u be continuous in D. If $A \cdot B$ is nonnegative in Ω and u satisfies the inequality*

$$u(x,t) \le f(x,t) + A(x,t) \int_0^t \int_a^b B(y,s) u(y,s)\, ds\, dt, \tag{6}$$

then

$$u(x,t) \leq f(x,t) + A(x,t) \int_0^t \int_a^b B(y,s) \times \exp\left[\int_s^t \int_a^b A(z,\tau) B(z,\tau) \, dz \, d\tau \right] f(y,s) \, ds \, dt. \tag{7}$$

Proof. In this case,

$$k_0(x,t,y,s) = k(x,t,y,s) = A(x,t) B(y,s).$$

By virtue of (5), we get

$$\begin{aligned} k_1(x,t,y,s) &= \int_s^t \int_a^b A(x,t) B(z,\tau) A(z,\tau) B(y,s) \, dz \, d\tau \\ &= A(x,t) B(y,s) \cdot L(t), \end{aligned}$$

where

$$L(t) = \int_s^t M(\tau) \, d\tau, \qquad M(\tau) = \int_a^b A(z,\tau) B(z,\tau) \, dz$$

and

$$L(s) = 0, \qquad L'(t) = M(t).$$

Similarly, we have

$$\begin{aligned} k_2(x,t,y,s) &= \int_s^t \int_a^b A(x,t) B(z,\tau) A(z,\tau) B(y,s) L(\tau) \, d\tau \, dz \\ &= A(x,t) B(y,s) \int_s^t M(\tau) L(\tau) \, d\tau \\ &= A(x,t) B(y,s) \int_s^t L'(t) L(\tau) \, d\tau \\ &= A(x,t) B(y,s) \int_s^t \frac{d}{d\tau} \left[\frac{L^2(\tau)}{2!} \right] d\tau \\ &= A(x,t) B(y,s) \frac{L^2(t)}{2!}. \end{aligned}$$

By induction, we obtain

$$k_n(x,t,y,s) = A(x,t) B(y,s) \frac{L^n(t)}{n!}.$$

Next, from (4) it follows that

$$\begin{aligned} r(x,t,y,s) &= A(x,t)B(y,s)\sum_{n=0}^{\infty}\frac{L^n(t)}{n!} \\ &= A(x,t)B(y,s)\exp[L(t)] \\ &= A(x,t)B(y,s)\exp\left[\int_s^t\int_a^b A(z,\tau)B(z,\tau)\,dz\,d\tau\right]. \end{aligned}$$

Using Theorem 1, the proof is finished. ∎

LEMMA 1. *If h is continuous in D, then*

$$1+\int_0^t\int_a^b h(y,s)\exp\left[\int_s^t\int_a^b h(z,\tau)\,dz\,d\tau\right]dy\,ds = \exp\left[\int_0^t\int_a^b h(y,s)\,dy\,ds\right].$$

Proof. Introduce the notation

$$\int_a^b h(y,s)\,dy = H(s), \qquad \int_0^t H(s)\,ds = \chi(t).$$

Then

$$\chi'(t) = h(t), \qquad \chi(0) = 0.$$

Notice that

$$\begin{aligned} 1 &+ \int_0^t\int_a^b h(y,s)\exp\left[\int_s^t\int_a^b h(z,\tau)\,dz\,d\tau\right]dy\,ds \\ &= 1+\int_0^t H(s)\exp\left[\int_s^t H(\tau)\,d\tau\right]ds \\ &= 1+\int_0^t \chi'(s)\exp[\chi(t)-\chi(s)]\,ds \\ &= 1+\exp\chi(t)\int_0^t\chi'(s)\exp[-\chi(s)]\,ds \\ &= 1+\exp\chi(t)\left\{\exp[-\chi(s)]\big|_0^t\right\} \\ &= 1-\exp\chi(t)\exp[-\chi(t)]+\exp\chi(t) \\ &= \exp\chi(t) = \exp\left[\int_0^t H(s)\,ds\right] = \exp\left[\int_0^t\int_a^b h(y,s)\,dy\,ds\right]. \quad ∎ \end{aligned}$$

COROLLARY 1. *If the assumptions of Theorem* 2 *are satisfied, then*

$$u(x,t) \le F(t)\left[1 + A(x,t)\int_0^t\int_a^b B(y,s) \times \exp\left[\int_s^t\int_a^b A(z,\tau)B(z,\tau)\,dz\,d\tau\right]dy\,ds\right], \tag{8}$$

where

$$F(t) = \sup\{f(x,s): a \le x \le b, 0 \le s \le t\}.$$

Remark 1. If $A(x,t) = 1$, we get an analogue of the Gronwall inequality

$$u(x,y) \le F(t)\exp\left[\int_0^t\int_a^b B(y,s)\,dy\,ds\right] \tag{9}$$

(it suffices to use Lemma 1).

Remark 2. Moreover, if $f(x,t) = c$ or f is bounded in D ($|f(x,t)| \le c$), then the inequality

$$u(x,t) \le f(x,t) + \int_0^t\int_a^b B(y,s)u(y,s)\,dy\,ds \tag{10}$$

implies

$$u(x,t) \le C\exp\left[\int_0^t\int_a^b B(y,s)\,dy\,ds\right]. \tag{11}$$

THEOREM 3. *Let the assumptions of Theorem* 2 *be fulfilled. If u satisfies inequality* (6), *then the following inequality*:

$$u(x,t) \le H(x,t)\exp\left[\int_0^t\int_a^b M(y,s)B(y,s)\,dy\,ds\right] \tag{12}$$

holds, where

$$H(x,t) = \max\{A(x,t), f(x,t)\} \ne 0.$$

Proof. Inequality (6) leads to

$$u(x,t) \le H(x,t)\left[1 + \int_0^t\int_a^b B(y,s)u(y,s)\,dy\,ds\right]$$

or

$$\frac{u(x,t)}{H(x,t)} \le 1 + \int_0^t\int_a^b H(y,s)B(y,s)\frac{u(y,s)}{H(y,s)}\,dy\,ds.$$

Applying Remark 2 with $c = 1$, we get

$$\frac{u(x,t)}{H(x,t)} \le \exp\left[\int_0^t \int_a^b H(y,s) B(y,s)\, dy\, ds\right]$$

and (12). ∎

COROLLARY 2. *If the assumptions of Theorem* 3 *are satisfied, then inequality* (12) *leads to*

$$u(x,t) \le A(x,t) \exp\left[\int_0^t \int_a^b A(y,s) B(y,s)\, dy\, ds\right], \quad \text{as } f(x,t) \le A(x,t) \ne 0, \tag{13}$$

or

$$u(x,t) \le f(x,t) \exp\left[\int_0^t \int_a^b f(y,s) B(y,s)\, dy\, ds\right], \quad \text{as } 0 \le f(x,t) < A(x,t). \tag{14}$$

THEOREM 4. *Suppose that the assumptions of Theorem* 2 *are fulfilled. If* $A(x,t) \ne 0$, *then inequality* (6) *implies*

$$u(x,t) \le \Phi(t) A(x,t) \exp\left[\int_0^t \int_a^b A(y,s) B(y,s)\, dy\, ds\right], \tag{15}$$

where

$$\Phi(t) = \sup\left\{\frac{f(x,s)}{A(x,s)} : a \le x \le b, 0 \le s \le t\right\}.$$

Proof. Inequality (6) can be written in the form

$$\frac{u(x,t)}{A(x,t)} \le \frac{f(x,t)}{A(x,t)} + \int_0^t \int_a^b A(y,s) B(y,s) \frac{u(y,s)}{A(y,s)}\, dy\, ds.$$

By virtue of Remark 1, we get

$$\frac{u(x,t)}{A(x,t)} \le \Phi(t) \exp\left[\int_0^t \int_a^b A(y,s) B(y,s)\, dy\, ds\right],$$

which concludes the proof. ∎

THEOREM 5. *Let f and k be continuous functions in D and Ω, respectively. If k is nonnegative and satisfies in Ω the condition*

$$k(x,t,y,s) \le K(y,s), \tag{16}$$

and a continuous function u satisfies inequality (2), *then*

$$u(x,t) \leq F(t)\exp\left[\int_0^t\int_a^b K(y,s)\,dy\,ds\right].$$

Proof. Applying Remark 1 to the inequality

$$u(x,t) \leq f(x,t) + \int_0^t\int_a^b K(y,s)u(y,s)\,dy\,ds,$$

the proof is finished. ∎

THEOREM 6. *Suppose that the assumptions of Theorem* 5 *are satisfied and the condition* (16) *is replaced by*

$$k(x,t,y,s) < N(x,t). \tag{17}$$

Then

$$u(x,t) \leq N^*(t)N(x,t)\exp\left[\int_0^t\int_a^b N(y,s)\,dy\,ds\right],$$

where

$$N^*(t) = \sup\left\{\frac{f(x,s)}{N(x,s)} : a \leq x \leq b,\ 0 \leq s \leq t\right\}.$$

Proof. By (17) from inequality (2) it follows that

$$u(x,t) \leq f(x,t) + N(x,t)\int_0^t\int_a^b u(y,s)\,dy\,ds$$

and

$$\frac{u(x,t)}{N(x,t)} \leq \frac{f(x,t)}{N(x,t)} + \int_0^t\int_a^b N(y,s)\frac{u(y,s)}{N(y,s)}\,dy\,ds.$$

Using Remark 1, we get the inequality

$$\frac{u(x,t)}{N(x,t)} \leq N^*(t)\exp\left[\int_0^t\int_a^b N(y,s)\,dy\,ds\right],$$

which concludes the proof. ∎

Remark 3. If, in (2), $k(x,t,y,s) \leq A(x,t)B(y,s)$, $A(x,t) \neq 0$, then we get inequality (6), which leads to (15).

Remark 4. The above results are true for Volterra–Fredholm inequalities

$$u(x,t) \le f(x,t) + \int_0^t \int_G k(x,t,y,s) u(y,s)\, dy\, ds, \tag{18}$$

where G is a certain compact subset of R^K.

Remark 5. The results presented here can be extended to the class L^2.

3. SOME APPLICATIONS

In this section, we present some applications of the above results to study the boundedness, stability, and uniqueness of the solutions of certain integral equations, their systems, and initial boundary problems for parabolic partial differential equations.

EXAMPLE 1. Consider the following nonlinear integral equation of the Volterra–Fredholm type:

$$u(x,t) = f(x,t) + \int_0^t \int_a^b K[x,t,y,s,u(y,s)]\, dy\, ds, \tag{19}$$

with assumptions:

(1) f and K are continuous in D and

$$\Theta = \{(x,t,y,s,u) \colon a \le x,\ y \le b,\ 0 \le s \le t < \infty,\ |u| < \infty\},$$

respectively,

(2) $$|K[x,t,y,s,u]| \le B(y,s)|u| \qquad \text{in } \Omega,$$

(3) $$\begin{aligned}|K(x_1,t_1,y_1,s_1,u_1) - K(x_2,t_2,y_2,s_2,u_2)|& \\ \le B(y,s)|u_1 - u_2| \qquad &\text{in } \Omega,\end{aligned}$$

where B is continuous and integrable in D.

Notice that from (19) we get the inequality

$$|(u,t)| \le |f(x,t)| + \int_0^t \int_a^b B(y,s)|u(y,s)|\, dy\, ds. \tag{20}$$

Applying Remark 1, we have

$$|u(x,t)| \le \Psi(t) \exp\left[\int_0^t \int_a^b B(y,s)\, dy\, ds\right], \tag{21}$$

where

$$\Psi(t) = \sup\{|f(x,t)| : a \le x \le b, 0 \le s \le t\}.$$

In this way the following result holds.

PROPOSITION 1. *If assumptions* (1) *and* (2) *of Example* 1 *are satisfied and* Ψ *is bounded in* $I = [0, \infty)$, *then a solution of* (19) *is bounded in* D.

Further, we can prove the stability and uniqueness of solutions of (19).

PROPOSITION 2. *If assumptions* (1) *and* (3) *of Example* 1 *are satisfied, then* (19) *has at most one solution, which is stable*.

Proof. Let u_1 and u_2 be the solutions of (19) corresponding to free terms f_1, f_2, respectively, such that $|f_1(x,t) - f_2(x,t)| < \varepsilon$ for arbitrary $\varepsilon > 0$.

Then, applying assumption (3) of Example 1 to (19), we get

$$\begin{aligned}|u_1(x,t) - u_2(x,t)| \le& |f_1(x,t) - f_2(x,t)| \\ &+ \int_0^t \int_a^b B(y,s)|u_1(y,s) - u_2(y,s)|\,dy\,ds \\ \le& \varepsilon + \int_0^t \int_a^b B(y,s)|u_1(y,s) - u_2(y,s)|\,dy\,ds.\end{aligned}$$

Using Remark 2, we obtain the inequality

$$|u_1(x,t) - u_2(x,t)| \le \varepsilon \exp\left[\int_0^t \int_a^b B(y,s)\,dy\,ds\right],$$

which gives the stability result.

The uniqueness of solutions of (20) is proved, because $f_1(x,t) = f_2(x,t)$. Then

$$|u_1(x,t) - u_2(x,t)| \le 0,$$

i.e.,

$$u_1(x,t) = u_2(x,t) \qquad \text{in } D. \quad \blacksquare$$

EXAMPLE 2. Now consider the following system of integral equations of Volterra–Fredholm type:

$$u_i(x,t) = f_i(x,t) + \sum_{j=1}^{m} \int_0^t \int_a^b k_{ij}(x,t,y,s) u_j(y,s)\,dy\,ds, \tag{22}$$

where f_i, $i = 1, 2, \ldots, m$, and k_{ij}, $i, j = 1, 2, \ldots, m$, are continuous in D and Ω, respectively.

Introducing the following notation:

$$\sum_{i=1}^{m} \left| u_i(x,t) \right| = u(x,t), \qquad \sum_{i=1}^{m} \left| f_i(x,t) \right| = f(x,t),$$

$$\sum_{i=1}^{m} \max_{1 \le j \le m} \left| k_{ij}(x,t,y,s) \right| \le B(y,s) \qquad \text{in } \Omega,$$

we get the inequality

$$u(x,t) \le f(x,t) + \int_0^t \int_a^b B(y,s) u(y,s)\, dy\, ds. \tag{23}$$

By virtue of Remark 1, we obtain

$$u(x,t) \le \Phi(t) \exp\left[\int_0^t \int_a^b B(y,s)\, dy\, ds \right], \tag{24}$$

where

$$\Phi(t) = \sup\{ |f(x,s)| : a \le x \le b, 0 \le s \le t \}.$$

From the above considerations the bounds of solutions of system (22) follow.

PROPOSITION 3. *Let f_i, $i = 1, 2, \dots, m$, be continuous in D and k_{ij}, $i, j = 1, 2, \dots, m$, be continuous in Ω, such that*

$$\sum_{j=1}^{m} \max_{1 \le j \le m} \left| k_{ij}(x,t,y,s) \right| \le B(y,s),$$

where B is continuous and integrable in D.

If Φ is bounded in $I = [0, \infty)$, then a solution $\{u_i(x,t)\}$, $i = 1, 2, \dots, m$, of system (22) *is bounded in D and an estimate is defined by* (24).

Remark 6. If f is bounded in D ($|f(x,t)| \le C$), then the bounded solution of system (22) is estimated by the inequality

$$\sum_{i=1}^{m} \left| u_i(x,t) \right| \le C \exp\left[\int_0^t \int_a^b B(y,s)\, dy\, ds \right].$$

PROPOSITION 4. *If the assumptions of Proposition* 3 *are satisfied, then system* (22) *has at most one solution, which is stable.*

Proof. It follows from the inequality

$$|u(x,y) - u^*(x,t)| \le \varepsilon + \int_0^t \int_a^b B(y,s) |u(y,s) - u^*(y,s)| \, dy \, ds,$$

which implies

$$|u(x,y) - u^*(x,t)| \le \varepsilon \exp\left[\int_0^t \int_a^b B(y,s) \, dy \, ds \right]$$

if

$$|u(x,y) - u^*(x,t)| < \varepsilon. \quad \blacksquare$$

EXAMPLE 3. Some initial-boundary-value problems for partial differential equations of the parabolic type (Fourier problems) are reducible to the Volterra–Fredholm integral equation

$$u(x,t) = f(x,t) + \int_0^t \int_G k(x,t,y,s) u(y,s) \, dy \, ds, \tag{25}$$

where G is a compact subset of R^n and f depends on the given initial and boundary conditions.

PROPOSITION 5. *If f and k are continuous in $G \times I$ and $(G \times I)^2$, respectively, such that*

$$|k(x,t,y,s)| \le B(y,s),$$

where B is continuous and integrable in $G \times I$, then a solution of (25) *is stable.*

Moreover, if f is bounded, then the solution is bounded, too.

Proof. It is clear that for $|f_1(x,t) - f_2(x,t)| < \varepsilon$ we get

$$|u_1(x,t) - u_2(x,t)| \le \varepsilon + \int_0^t \int_G B(y,s) |u_1(y,s) - u_2(y,s)| \, dy \, ds.$$

Using Remark 2, we obtain the estimate

$$|u_1(x,t) - u_2(x,t)| \le \varepsilon \exp\left[\int_0^t \int_G B(y,s) \, dy \, ds \right],$$

which proves the stability of the solution of (25).

The boundedness of the solution of (25) follows from the inequality

$$|u(x,t)| \le |f(x,t)| + \int_0^t \int_G B(y,s)|u(y,s)|\,dy\,ds,$$

which implies

$$|u(x,t)| \le C \exp\left[\int_0^t \int_G B(y,s)\,dy\,ds\right],$$

because f is bounded ($|f(x,t)| \le C$). ∎

REFERENCES

1. D. Bainov and P. Simeonov, "Integral Inequalities and Applications," p. 244, Kluwer Academic, Dordrecht/Boston, 1992.
2. B. K. Bondge, B. G. Pachpatte, and W. Walter, On generalized Wendroff type inequalities and their applications, *Nonlinear Anal.* **4** (1980), 491–495.
3. A. Corduneanu, A note on Gronwall inequality in two independent variables, *J. Integral Equations* **4** (1982), 271–276.
4. L. Hącia, On integral inequalities in two variables and their applications, to appear.
5. L. Hącia, On solving Volterra–Fredholm integral equation, to appear.